A simple proof of formula for Euler Characteristic

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I present a simple proof of Euler's formula. Obviously, there are many proofs out there, and there might be one exactly like this one. None-the-less, this is a very revealing proof about what Euler's formula follows from.

Given a $k-$ dimensional simplicial complex K, denote the set of $i-simplices$ by V_i. Let f_i denote the number of $i-$simplices and b_i denote the i^{th} Betti number. I will use the following fact, which I state without proof.

Fact I Given a subcomplex K' which contains the boundary of an $i-$simplex $\sigma^i \notin K'$, adding σ^i to K' either kills an $i-1$ cycle or creates an i cycle.

Theorem 0.1

$$\sum_{i=0}^{k} (-1)^i f_i = \sum_{i=0}^{k} (-1)^i b_i$$

Proof Lets start constructing our complex from scratch, with $K' = \varnothing$. First, throw in all the 0-simplices. Then $K' = V_0$, and

$$f_0 = b_0$$

Now, throw in all the 1-simplices. Suppose c_1 of them kill 0-cycles and c_2 of them create 1-cycles. For $K' = V_0 \cup V_1$, we have

$$\begin{aligned} b_0 &= f_0 - c_1 \\ b_1 &= c_2 \\ c_1 + c_2 &= f_1 \end{aligned}$$

Subtracting the second and third equation from the first, we get

$$b_0 - b_1 = f_0 - f_1 \tag{1}$$

Since this can go on for a while, I will use induction. For $m \geq 1$, we have

$$(-1)^m b_m = \sum_{i=0}^{m} (-1)^i f_i - \sum_{i=0}^{m-1} (-1)^i b_i \tag{2}$$

We then throw in all the $m+1$ simplices, c_1 of which kill $m-$cycles and c_2 create $m+1$ cycles. We have

$$\begin{aligned}
(-1)^m b_m &= \sum_{i=0}^{m} (-1)^i f_i - \sum_{i=0}^{m-1} (-1)^i b_i + (-1)^{m+1} c_1 \\
(-1)^{m+1} b_1 &= (-1)^{m+1} c_2 \\
(-1)^{m+1} (c_1 + c_2) &= (-1)^{m+1} f_{m+1}
\end{aligned}$$

Adding the second and third equation to the first, we get

$$\sum_{i=0}^{m+1} (-1)^i f_i = \sum_{i=0}^{m+1} (-1)^i b_i$$

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Conclusion The formula follows entirely from the fact that adding σ^i either kills an $i-1$ cycle or creates an i cycle.